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On Boundary Conditions for Hyperbolic Difference Schemes*

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We describe an investigation, largely experimental, to determine stable approximations for the boundary conditions of hyperbolic systems. We are concerned with the fully discrete leapfrog scheme, and also with the method of lines which uses a finite difference approximation in space and an ODE solver in time. We are especially interested in schemes which have fourth order accuracy. Our discussion concerns problems in only one space dimension. We describe a modification of the scheme developed by Oliger [7] to stabilize the leapfrog scheme when a fourth order spatial difference approximation is used. We analyze boundary approximations for the method of lines by a numerical study of the eigenvalues and norm of the matrix for the semi-discrete system. These are checked by integrations of the system using an ODE solver. We also study the use of boundary conditions which are differentiated in time. This is done in order to obtain a system to which an ODE solver can be conveniently applied.

1. A DuFort Frankl stabilizer

We are first concerned with the simple hyperbolic equation

$$u_t + u_x = 0 \tag{1}$$

with the initial-boundary data

$$\begin{aligned} u(x, 0) &= f(x) \qquad 0 \leqslant x \leqslant 1 \\ u(0, t)\, &g(t) \qquad 0 \leqslant t \end{aligned}$$

The method of Gustafsson et. al. [19] can be used to show that the following version of the leapfrog scheme is unstable due to the boundary approximation (we use the notation $u_j{}^n = u(x_j, t_n)$).

$$\begin{aligned} U_j^{n+1} &= U_j^{n-1} - \lambda(U_{j+1}^n - U_{j-1}^n) \qquad 0 < j < J \\ U_0^{n+1} &= g(t_{n+1}) \\ U_J^{n+1} &= U_J^{n-1} - 2\lambda(U_J{}^n - U_{J-1}^n) \\ x_j &= j\,\Delta x = j/J \qquad \lambda = \Delta t/\Delta x \end{aligned} \tag{2}$$

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They show that replacement of the downstream boundary by the following approximation yields a stable scheme.

$$U_J^{n+1} = U_J{}^n - \lambda(U_J{}^n - U_{J-1}^n) \tag{3}$$

Oliger [7] has shown how to obtain a stable "leapfrog" scheme with accuracy $O(\Delta t^2 + \Delta x^4)$ for this problem. A generalization of (2) using fourth order spatial differences in the interior and third order one-sided differences near the boundary is unstable. Use of a first order one-sided difference in time similar to (3) would reduce the accuracy. Oliger uses a time average at points near the boundary in order to stabilize the scheme, namely the following.

$$\begin{aligned}
U_j^{n+1} &= U_j^{n-1} - \frac{\lambda}{6}(U_{j-2}^n - 8U_{j-1}^n + 8U_{j+1}^n - U_{j+2}^n) \qquad 2 \leqslant j \leqslant J-2 \\
U_1^{n+1} &= U_1^{n-1} - \frac{\lambda}{3}\left(-2U_0{}^n - \frac{3}{2}(U_1^{n+1} + U_1^{n-1}) + 6U_2{}^n - U_3{}^n\right) \\
U_{J-1}^{n+1} &= U_{J-1}^{n-1} - \frac{\lambda}{3}\left(U_{J-3}^n - 6U_{J-2}^n + \frac{3}{2}(U_{J-1}^{n+1} + U_{J-1}^{n-1}) + 2U_J{}^n\right) \\
U_J^{n+1} &= U_J^{n-1} - \frac{\lambda}{3}\left(-2U_{J-3}^n + 9U_{J-2}^n - 18U_{J-1}^n + \frac{11}{2}(U_J^{n+1} + U_J^{n-1})\right)
\end{aligned} \tag{4}$$

Application of this time averaging to the second order scheme (2) also yields a stable scheme whose boundary approximation can be written in the form

$$\begin{aligned}
U_J^{n+1} &= U_J^{n-1} - 2\lambda(\tfrac{1}{2}(U_J^{n+1} + U_J^{n-1}) - U_{J-1}^n) \\
&= U_J^{n-1} - 2\lambda(U_J{}^n - U_{J-1}^n) - \lambda(U_J^{n+1} - 2U_J{}^n + U_J^{n-1})
\end{aligned} \tag{5}$$

This approximation was used by Elvius and Sundström [4]. This can be regarded as the addition of a stabilizing term to the boundary approximation of equation (2). This is similar to the DuFort Frankl scheme for the heat equation,

$$U_t = U_{xx}$$

which can be written as

$$U_j^{n+1} = U_j^{n-1} + \frac{2\Delta t}{\Delta x^2}(U_{j+1}^n - 2U_j{}^n + U_{j-1}^n) - \frac{2\Delta t}{\Delta x^2}(U_j^{n+1} - 2U_j{}^n + U_j^{n-1})$$

Gottlieb and Gustafsson [17] have studied the application of this stabilizing term to the heat equation.

All this suggests that we can regard the time-averaging of Sundström and Oliger as a DuFort Frankl type of stabilizing term. Therefore we consider the following leapfrog schemes for equation (1). The first has second order accuracy in the mesh interior

$$\begin{aligned}
U_0^{n+1} &= g_0(t_{n+1}) \\
U_j^{n+1} &= U_j^{n-1} - 2\Delta t\, \delta_j^{(2)}(U^n) - \gamma_j \lambda \mu (U_j^{n+1} - 2U_j{}^n + U_j^{n-1})
\end{aligned}$$

The parameter μ can be varied in order to determine the minimum stabilization term (see tables I and II).

The above symbols are defined by

$$
\begin{aligned}
\delta_j^{(2)}(U^n) &= (U_{j+1}^n - U_{j-1}^n)/(2\Delta x) \qquad 1 \leqslant j < J \\
\delta_J^{(2)}(U^n) &= (U_J^n - U_{J-1}^n)/\Delta x \\
\gamma_j &= 0 \qquad \text{for} \quad 1 \leqslant j < J \\
\gamma_J &= 1 \\
\lambda &= \Delta t/\Delta x
\end{aligned}
\tag{6}
$$

The second scheme uses a fourth order spatial approximation.

$$U_j^{n+1} = U_j^{n-1} - 2\Delta t\, \delta_j^{(4)}(U^n) - \gamma_j \lambda \mu (U_j^{n+1} - 2U_j^n + U_j^{n-1}) \tag{7}$$

where

$$\delta_j^{(4)}(U^n) = (U_{j-2}^n - 8U_{j-1}^n + 8U_{j+1}^n - U_{j+2}^n)/(12\Delta x) \qquad 2 \leqslant j \leqslant J-2$$

The parameter γ_j is set to correspond to the averaging scheme given in (4). Thus

$$
\begin{aligned}
&\gamma_j = 0 \text{ for } 2 \leqslant j \leqslant J-2 \\
&\gamma_1 = \gamma_{J-1} = \tfrac{1}{2} \\
&\gamma_J = 11/16
\end{aligned}
$$

The operators $\delta_j^{(4)}$ have third order accuracy near the boundary. No operator is needed at $j = 0$ for equation (1). The parameter μ can be varied. This parameter will determine the stability and, to some extent, the accuracy of the scheme. The main advantage of the DuFort Frankl term over the averaging is in its application to a nonlinear equation. For example, consider an equation written in conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = 0$$

Application of the averaging scheme to this equation requires the solution of a nonlinear equation. We have not tested the DuFort Frankl stabilization on a nonlinear system of equations. However, it did seem to work on a simple nonlinear hyperbolic equation with a single unknown. We regard the parameter μ as a scaling factor. In the case of a nonlinear system, this parameter should probably include the norm of the Jacobian matrix of the system as a factor.

We test the difference schemes (6) and (7) by application to problem (1), except we use the interval $0 \leqslant x \leqslant 0.5$. The solution is taken to be

$$u(x, t) = \sin 2\pi(x - t)$$

Therefore, the initial boundary data is

$$u(x, 0) = \sin 2\pi x$$
$$u(0, t) = -\sin 2\pi t$$

The mesh is given by

$$x_j = j\Delta x = j/(2J) \qquad 0 \leqslant j \leqslant J$$

To test the stability of the second order scheme (6) we use $J = 10$, and run out to a time $t = 160$. We abort the run and call the scheme unstable if

$$\max_j | U_j^n | = \| U^n \|_\infty > 2$$

A similar test used for the fourth order scheme (7) except the test was terminated at $t = 80$. The results are shown in table I.

The accuracy of the solution, at least in our examples, did not seem to depend much on μ. With $J = 10$ and $\lambda = .2$ the error from the fourth order scheme is shown in table II.

TABLE I

Stability of Schemes (6) and (7)[a]

Stability of the second order scheme (6)				
$\lambda = 0.9$	$\mu = 0.6$ unstable	$\mu = 0.7$ stable	$\mu = 1.8$ stable	$\mu = 2.0$ unstable
$\lambda = 0.5$	$\mu = 0.5$ unstable	$\mu = 0.6$ stable	$\mu = 7.0$ stable	$\mu = 8.0$ unstable
Stability of the fourth order scheme (7)				
$\lambda = 0.4$	$\mu = 0.5$ unstable	$\mu = 0.6$ stable	$\mu = 4.0$ stable	$\mu = 5.0$ unstable
$\lambda = 0.2$	$\mu = 0.4$ unstable	$\mu = 0.5$ stable	$\mu = 20.$ stable	$\mu = 25.$ unstable

[a] Scheme (6) ran to $t = 160$, scheme (7) to $t = 80$. $J = 10$ with solution $u = \sin 2\pi(x - t)$, $0 \leqslant x \leqslant 0.5$.

TABLE II

The Error for Scheme (7) with $J = 10$, $\lambda = 0.2$ for the Solution $u = \sin 2\pi(x - t)$ at $t = 1.0$

μ	0.7	1.0	1.5
Error	5.3(−3)	5.8(−3)	6.8(−3)

We will also use the following simple hyperbolic system as a test case.

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \frac{\partial u_2}{\partial x} \qquad 0 \leqslant x \leqslant a \\ \frac{\partial u_2}{\partial t} &= \frac{\partial u_1}{\partial x} \qquad 0 \leqslant t \end{aligned} \tag{8}$$

The characteristic variables and equations are

$$\begin{aligned} w_1 &= u_1 - u_2 \qquad \frac{\partial w_1}{\partial t} = -\frac{\partial w_1}{\partial x} \\ w_2 &= u_1 + u_2 \qquad \frac{\partial w_2}{\partial t} = \frac{\partial w_2}{\partial x} \end{aligned} \tag{9}$$

In all of the experiments with the system (8) we choose the initial-boundary data so that the solution is

$$\begin{aligned} w_1(x, t) &= \sin 2\pi(x - t) \\ w_2(x, t) &= \sin 2\pi(x + t) \end{aligned} \tag{10}$$

The corresponding solution in terms of $\underline{u}$ is given by

$$\begin{aligned} u_1 &= \tfrac{1}{2}(w_1 + w_2) \\ u_2 &= \tfrac{1}{2}(w_2 - w_1) \end{aligned} \tag{11}$$

2. Boundary conditions for the method of lines

The method of lines is based on the system of ordinary differential equations (DE) obtained when the spatial derivatives are replaced by finite differences. This system of ODE is then solved by an ODE solver. The method has been applied to partial differential equations by many people including Sincovec and Madsen [11], Carver [2], Loeb [6], Bowen [1], Hastings [12] and others. There are several excellent ODE solvers available for use with this method. The method of lines is certain to be stable when it is applied to symmetric hyperbolic systems with constant coefficients and periodic boundary conditions of the form

$$\frac{\partial \underline{u}}{\partial t} + A\frac{\partial \underline{u}}{\partial x} = 0 \qquad \underline{u} = \underline{u}(x, t) \tag{12}$$

A reasonable finite difference scheme will result in a system of ODE whose matrix M is skew symmetric

$$\underline{U}' = M\underline{U} + \underline{G}(t) \tag{13}$$

Note that $G(t) \equiv 0$ in the case of periodic boundary conditions. The eigenvalues of the matrix M are pure imaginary and the eigenvectors are orthogonal. Note that the matrix A is assumed to be symmetric. Therefore the norm of the exponential matrix is bounded uniformly independent of the mesh spacing used in the spatial discretization. The solution of the system of ODE (13) can be written in the form

$$U(t) = e^{Mt}\underline{U}(0) + \int_0^t e^{M(t-\tau)}\underline{G}(\tau)\, d\tau$$

Therefore the difference scheme will be stable. This is true even if the Euler method is used in the ODE solver. The Euler method with a fixed ratio $\lambda = \Delta t/\Delta x$ is unconditionally unstable. However the ODE solver will adjust the step size to produce the desired error in the solution of the system of ODE. Of course, this would be expensive. The ODE solver should probably be based on a time discretization which is stable when applied to a hyperbolic system. Numerical experiments [14] indicate that the method of lines may not be as efficient as the Kriess-Oliger version of the leapfrog which uses fourth order differences in space and a second difference in time. However the difference is not great and the method of lines is easier to use since it is not necessary to choose a value of Δt in order to obtain optimal performance. Our purpose in this section is to study the stability of the method of lines for non-periodic boundary conditions.

We take an experimental approach to this problem, which is to code the difference schemes and test them. We also study the properties of the matrix M in the equation for the method of lines (13). We compute the eigenvalues of M, although these do not determine stability unless we also know something about the eigenvectors of M. We also compute the norm of the exponential matrix $\exp(M)$, although what we really need is a uniform bound on the norm of $\exp(Mt)$. We look at four schemes for the simple hyperbolic equation (1).

A. *An inconsistent scheme.* We first look at a scheme which we know is unstable. This scheme uses one-sided differences at both boundaries instead of setting the solution at the inflow boundary. This scheme is defined by

$$U_j'(t) = -\delta_j^{(2)}(U(t)) \qquad 0 \leqslant j \leqslant J \tag{14}$$

where the operator is as given in (6), with

$$\delta_0^{(2)} = (U_1(t) - U_0(t))/\Delta x$$

B. *A consistent second order scheme.* This scheme is also applied to equation (1). It is defined by

$$\begin{aligned} U_0(t) &= -\sin 2\pi t \\ U_j'(t) &= -\delta_j^{(2)}(U(t)) \qquad 1 \leqslant j \leqslant J \end{aligned} \tag{15}$$

This scheme is only first order at the boundary, but the overall accuracy is second order.

C. *A fourth order scheme with third order boundary approximation.* This scheme applies to (1) and is defined by

$$\begin{aligned} U_0(t) &= -\sin 2\pi t \\ U_j'(t) &= -\delta_j^{(4)}(U(t)) \qquad 1 \leqslant j \leqslant J \end{aligned} \tag{16}$$

The operator $\delta_j^{(4)}$ is defined in (7). It is fourth order in the interior and third order at $j = 1, J-1$, and J.

D. *A fourth order scheme with a fourth order boundary approximation.* This scheme is the same as the preceeding except that a fourth order five point operator is used at the points $j = 1, J-1$, and J. The operator is defined as follows at these points.

$$\begin{aligned} \delta_1^{(4)}(U) &= (-6U_0 - 20U_1 + 36U_2 - 12U_3 + 2U_4)/(24\Delta x) \\ \delta_{J-1}^{(4)}(U) &= (-2U_{J-4} + 12U_{J-3} - 36U_{J-2} + 20U_{J-1} + 6U_J)/(24\Delta x) \\ \delta_J^{(4)}(U) &= (6U_{J-4} - 32U_{J-3} + 72U_{J-2} - 96U_{J-1} + 50U_J)/(24\Delta x) \end{aligned} \tag{17}$$

E. *A fourth order scheme with non-characteristic boundary approximation.* In this case we use the method of lines to solve the system (8). The scheme is the following where the operator $\delta^{(4)}$ is fourth order in the interior and third order near the boundary. It is the same operator as used in scheme C.

$$\begin{aligned} U_1'(0, t) &= 0. \\ U_1'(J, t) &= \pi(\cos 2\pi(a + t) - \cos 2\pi(a - t)) \\ U_1'(j, t) &= \delta_j^{(4)}(U_2(t)) \qquad 1 \leqslant j \leqslant J-1 \\ U_2'(j, t) &= \delta_j^{(4)}(U_1(t)) \qquad 0 \leqslant j \leqslant J \end{aligned} \tag{18}$$

Note that the boundary conditions are differentiated, that is we specify $U_1'(0, t)$ and not $U_1(0\ t)$. We will say more about this in the next section.

We analyze these schemes in two ways. The norm of the exponential matrix $\exp(M)$ is computed where M is the matrix of the system defined in (21). The summation

$$\sum_{k=0}^{N} M^k/k!$$

is terminated when $\| M^k \|_\infty/k! < 10^{-3}$. We use the maximum norm defined by

$$\max_i \sum_j | m_{ij} |$$

This is fairly expensive especially for the system (18) therefore we did not run too many experiments. This norm provides some insight into stability, but we really need a bound for $\| \exp(Mt)\|$ which is uniform in t and also uniform with respect to the order of M, that is, with respect to the mesh spacing.

The second part of the analysis involves solving the system with an ODE solver. In some cases (table IV) a Runge-Kutta-Fehlberg method taken from a report by Hull and Enright [5] is used. In other cases a ODE program based on the Adams multistep method obtained from Shampine and Gordon [9] is used.

The results giving the properties of the matrix M are in table III. We know scheme (A) can not be stable. However, the eigenvalues of M appear from these and other computations to be pure imaginary. If this is actually the case, then there must be an eigenvector deficiency, or the norm of the matrix of eigenvectors must be unbounded when the mesh spacing goes to zero. This is the only unstable scheme which does not have an eigenvalue with positive real part which we find rather interesting. The matrix M for scheme (A) appears to have a double or triple root at zero depending on odd or even J. We have shown the maximum real part of the eigenvalues as zero, but they can be around 10^{-4} because of rounding error in the triple root. For scheme (A) the instability is quite apparent in the norm of the exponential matrix. In the other cases this norm does not distinguish between stable and unstable schemes. We would expect this norm to be a poor indicator of stability or instability, unless it is extremely large.

TABLE III

Behavior of the Matrix M from (13)[a]

		λ_r	$\Vert e^M \Vert_\infty$
Inconsistent 2*nd*	5	0.0	50.
order scheme (A)	10	0.0	199.
$a = 1.0$	20	0.0	798.
Consistent 2*nd*	5	-0.26	1.7
order scheme (B)	10	-0.08	2.5
$a = 1.0$	20	-0.02	3.5
4th order with	5	-0.34	3.9
3rd order boundary (C)	10	-0.05	6.1
$a = 1.0$	20	$-9.8E\text{-}3$	8.7
4th order with	5	-0.07	6.7
4th order boundary (D)	10	0.22	11.1
$a = 1.0$	20	0.24	16.6
System (18) with non-	5	1.42	22.2
characteristic boundary	10	0.82	24.0
approx. (E) $a = 0.5$	20	0.57	—

[a] Here λ_r denotes the maximum over the real parts of the eigenvalues of M. The interval is $0 \leqslant x \leqslant a$.

We are indebted to one of the reviewers for this paper for the following enumeration of the eigenvalues of the difference scheme of (A). For odd J, the eigenvalues are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -\lambda_4 = 2i \sin(\pi/2J)$,... to $\lambda_J = -\lambda_{J+1} = 2i \sin((J-2)\pi/2J)$. For even J, they are $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = -\lambda_5 = 2i \sin(\pi/2J),..., \lambda_J = -\lambda_{J+1} = 2i \sin((J-2)\pi/2J)$.

The eigenvalues indicate that schemes (B) and (C) are stable whereas (D) and (E) are unstable. This is confirmed by integration of the equations using an ODE solver (see table IV). The instability in (D) apparently requires a small error tolerance in order to solve the equations accurately for large t. This same phenomenon occurs with the unstable scheme (18) for the system (8). This is illustrated in table V. Note the extremely small values of the error tolerance ϵ required to obtain an accurate solution at $t = 20$. We are surprised that it is possible to obtain such an accurate solution when the scheme has the strong instability indicated by an eigenvalue whose real part has the value 1.4. The integrations in table V do not show this strong instability especially if the system is integrated very accurately by using a small error tolerance ϵ.

TABLE IV

Error for Various Schemes Applied to (1)[a]

	J	$t = 1.0$	$t = 10.0$	$t = 20.0$	$t = 80.0$
Inconsistent scheme (A)	10	9.44	665.0	—	—
Second order scheme (B)	10	5.6(−2)	5.7(−2)	5.7(−2)	5.7(−2)
Fourth order scheme (C)	5	1.2(−2)	1.6(−2)	1.6(−2)	1.6(−2)
	10	4.3(−3)	4.6(−3)	5.0(−3)	4.9(−3)
	20	2.5(−4)	—	—	—
Fourth order including boundary (D)	5	3.2(−2)	2.5(−2)	2.5(−2)	2.5(−2)
	10	6.9(−4)	7.6(−3)	4.2(−1)	—
	20	4.4(−5)	—	—	—
Fourth order with differentiated boundary Eq. (22)	5	1.2(−2)	1.5(−2)	1.6(−2)	1.6(−2)
	10	4.3(−3)	4.3(−3)	4.3(−3)	4.3(−3)
	20	2.4(−4)		—	—

[a] Here $a = 0.5$ $(0 \leqslant x \leqslant a)$ and ϵ is chosen as required for the desired accuracy.

TABLE V

Error for the System (18) with Noncharacteristic Boundary Approximation (E)[a]

		Error (t)		
	ϵ	1.0	10.0	20.0
$J = 5$	1.0(−4)	1.2(−2)	1.1(−1)	2.0
	1.0(−6)	1.2(−2)	1.2(−1)	8.1(−1)
	1.0(−8)	1.2(−2)	1.2(−1)	2.9(−1)
$J = 10$	1.0(−4)	8.3(−4)	—	—

[a] Solved by Adams ODE solver where ϵ is the absolute error tolerance. Here $a = 0.5$.

However, the growth rate due to a perturbation in the initial data of the form $.1(-1)^j$ produces an eventual growth rate in the error between 0.4 and 1.8 which is in reasonable agreement with the maximum real part of the eigenvalue of 1.4 (when $J = 5$). The results for a stable scheme for the system (8) are given in the next section.

Perhaps it is unreasonable to draw any general conclusions from such limited experiments, but we will attempt to do so. It appears that boundary conditions for the method of lines may be slightly less troublesome than with a scheme which uses a fixed time step. The leapfrog scheme with fourth order spatial differences requires a time-average for stability which the method of lines does not require. Gottlieb [13] has had difficulty in obtaining a stable boundary approximation for a dissipative fourth order version of the Lax Wendroff scheme. The method of lines did not work with scheme (D). Also, the method of lines failed with the boundary approximation (18). So we conclude that the method of lines is slightly easier to stabilize than the leapfrog scheme, but the boundary approximation is still troublesome.

3. A DIFFERENTIATED BOUNDARY CONDITION FOR THE METHOD OF LINES

We want to apply existing ODE solvers, such as the RKF(Runge-Kutta-Fehlberg) of Watts and Shampine [15] or the ODE(Adams method) of Shampine and Gordon [9] to the systems obtained by spatial discretization of hyperbolic systems. If the system is derived in a direct way it may not be easy to apply an ODE solver. For example, consider the equation

$$\frac{\partial u}{\partial t} = -\cos(t)\frac{\partial u}{\partial x} \tag{19}$$

on the interval $0 \leqslant x \leqslant 1$. If $\cos(t) > 0$, then the semidiscrete approximation might be

$$\begin{aligned} &U_0(t) = g_0(t) \\ &U_j'(t) = -\cos(t)\,\delta_j^{(2)}(U(t)) \qquad 1 \leqslant j \leqslant J \end{aligned}$$

On the other hand if $\cos(t) < 0$, then we have

$$\begin{aligned} &U_j'(t) = -\cos(t)\,\delta_j^{(2)}(U(t)) \qquad 0 \leqslant j \leqslant J-1 \\ &U_J(t) = g_J(t). \end{aligned} \tag{20}$$

The boundary condition must be applied first on one side of the interval, and then on the other. The system of differential equations changes with time. This means that existing ODE solvers can not be used unless their coding is modified. This modification does not appear difficult in the case of an RKF code, but it does appear to be difficult for the multistep Adams codes. Therefore we differentiate the boundary conditions in order to obtain a uniform system of differential equations. The left boundary approximation for (19) thus becomes

$$U_0'(t) = \begin{cases} g_0'(t) & \text{if } \cos(t) \geqslant 0 \\ -\cos(t)\, \delta_0^{(2)}(U(t)) & \text{if } \cos(t) < 0 \end{cases}$$

There is a possible discontinuity in the right side of the differential equation, however there is always a differential equation for each variable. Madsen [16] uses such a differentiated boundary condition in a package for the solution of one dimensional PDE which is based on the method of lines. It has given good results.

F. *A scheme with differentiated boundary.* We first apply differentiated boundary conditions to a fourth order scheme for equation (1). The scheme is then

$$\begin{aligned} U_0'(t) &= -2\pi \cos 2\pi t \\ U_j'(t) &= -\delta_j^{(4)}(U(t)) \qquad 1 \leqslant j \leqslant J \end{aligned} \tag{22}$$

The operator $\delta_j^{(4)}$ is defined in (7). The results are given in table IV. There is clearly little difference between the scheme (C) with nondifferentiated boundary and this scheme (F).

Next we consider the use of differentiated boundary conditions for the system (8). We write the equations in terms of the original variables u_1 and u_2. The coding will be simplified if we avoid explicit use of the characteristic variables w_1 and w_2 at the boundary. However we must arrange the approximation to specify the inflow characteristic and compute the outflow characteristic from one-sided differences; that is, the boundary conditions must be given in terms of w_1 and w_2 in order that the scheme be stable. We first write the equations for the inflow characteristic at $x = 0$.

$$u_1 - u_2 = -\sin 2\pi t$$

and then differentiate to obtain

$$U_1'(0) - U_2'(0) = -2\pi \cos 2\pi t$$

The equation for the outflow characteristic is

$$U_1'(0) + U_2'(0) = \delta_0^{(4)}(U_1 + U_2) \tag{23}$$

The spatial derivatives in the equation have been approximated by one-sided differences of third order accuracy. Third order approximations are also used at the points adjacent to the boundary. The difference scheme in the interior is given below where the operator $\delta_j^{(4)}$ is defined by (7).

$$\begin{aligned} U_1'(t) &= \delta_j^{(4)}(U_2(t)) \\ U_2'(t) &= \delta_j^{(4)}(U_1(t)) \end{aligned} \tag{24}$$

At the boundary the two characteristic equations can be combined to obtain equations in the original variables u_1 and u_2.

$$\begin{aligned} U_1' &= (\delta_0^{(4)}(U_1 + U_2) - 2\pi \cos 2\pi t)/2 \\ U_2' &= (\delta_0^{(4)}(U_1 + U_2) + 2\pi \cos 2\pi t)/2 \end{aligned} \tag{25}$$

The result of solving these equations with the Adam's ODE solver is given in table VI. The method seems to work satisfactorily.

TABLE VI

Error for the Adam's Method ODE Solver Applied to the System (24) and (25)[a]

J	$t = 1.0$	$t = 10.0$	$t = 20.0$	$t = 80.0$
5	8.9(−3)	9.4(−3)	9.4(−3)	9.4(−3)
10	2.9(−3)	3.0(−3)	3.2(−3)	4.8(−3)
20	1.8(−4)	—	—	—

[a] Here $0 \leqslant x \leqslant 0.5$, ϵ set as required by desired accuracy.

It would certainly be desirable to express the boundary conditions directly in terms of the original variables U_1 and U_2 rather than in terms of the characteristic variables W_1 and W_2. However, this can result in an unstable difference scheme as Chu and Sereny [3] and Sundstrom [10] have shown. The use of non-characteristic variables in the boundary approximation does not always produce an unstable scheme [18, 4]. However, in the case of the simple system (8), the difference scheme is unstable if the boundary conditions are given in terms of the original variables u_1 and u_2 rather than in terms of the characteristic variables W_1 and W_2 as outlined above. In the case of the scond order leapfrog scheme this can be proven using the methods of Gustafsson, et. al. [19]. Our numerical experiments described above indicate that the same restriction applies to the method of lines. Gunzburger has found the same type of boundary instability is caused by the use of noncharacteristic variables in a finite element approximation of the hyperbolic system (8) [20].

Of course, experiments such as this based largely on linear equations do not prove anything. This is especially true if the methods are applied to nonlinear equations. We have tested the differentiated boundary condition for second and fourth order schemes on a few simple nonlinear equations including the shallow water equations without problems, however we have not yet run enough experiments to be able to report the results.

REFERENCES

1. S. BOWEN, "AMPCT, A Numerical Integration Routine for Systems of Stiff Differential Equations," Department of Meterorlogy, Univ. of Michigan, Report #033390-T, 1971.
2. M. CARVER, *in* "Advances in Computer Methods for Partial Differential Equations," (Vichnevetsky, Ed.), AICA, Rutgers Univ., 1975.
3. C. CHU AND A. SERENY, *J. Computational Phys.* **15** (1974), 476–491.
4. T. ELVIUS AND A. SUNDSTRÖM, *Tellus* **25** (1973), 32–156.
5. T. HULL AND W. ENRIGHT, "A Structure for Programs that Solve Ordinary Differential Equations," Technical Report #66, Department of Computer Science, Univ. of Toronto, 1974.
6. A. LOEB, *in* "Advances in Computer Methods for Partial Differential Equations" (Vichnevetsky, (Vichnevetsky, Ed.), AICA, Ruthers Univ., 1975.
7. J. OLIGER, *Math. Comp.* **28** (1974), 15–25.
8. A. SUNDSTRÖM, "Efficient Numerical Methods for Solving Wave Propagation Equations for Non-Homogeneous Media," C4576-A2, Swedish National Defense Research Institute, 1974.
9. L. SHAMPINE AND M. GORDON, "Computer Solution of Ordinary Differential Equations," Freeman, San Francisco, 1975.
10. A. SUNDSTRÖM, *J. Computational Phys.* **17** (1975), 450–454.
11. SINCOVEC AND N. MADSEN, "Software for Nonlinear Partial Differential Equations," Math. Software II Conf., Purdue University, 1974.
12. J. HASTINGS AND R. ROBLE, "A Technique for Solving the Coupled Time-Dependent Electron and Ion Energy Equations in the Ionosphere," NCAR manuscript, Boulder, Color. 1972.
13. D. GOTTLIEB AND E. TURKEL, "Dissipative Two-Four Methods for Time Dependent Problems," ICASE Rep. 75–22, NASA Langley, Hampton, V. 1975.
14. J. GARY, *J. Computational Phys.* **22** (1976), 131–149.
15. L. SHAMPINE AND H. WATTS, "RKF Code." Sandia Laboratories, Alburquerque, N.M.
16. N. MADSEN, private communication.
17. D. GOTTLIEB AND B. GUSTAFSSON, "Generalized DuFort Frankl Methods for Parabolic Initial-Boundary-Value Problems," ICASE Rep. 75-5, NASA–Langley, Hampton, V., 1975.
18. D. GOTTLIEB AND B. GUSTAFSSON, "On the Navier–Stokes Equations with Constant Total Temperature," ICASE Rep. 75–12, NASA–Langley, Hampton, V., 1975.
19. B. GUSTAFSSON, H. KREISS, AND A. SUNDSTROM, *Math. Comp.* **26** (1972), 649–685.
20. M. GUNZBURGER, "On the Stability of Galerkin Methods for Initial-Boundary Value Problems for Hyperbolic Systems," ICASE Report 76-3, ICASE, NASA–Langley Research Center, Hampton, V., 1976.